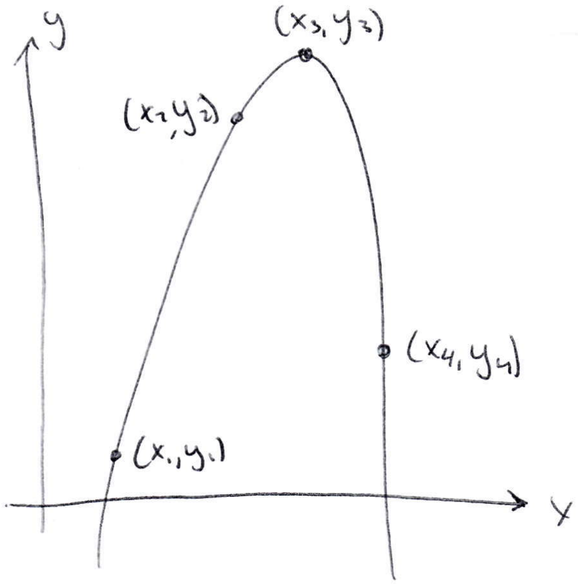
The tangent line to a graph

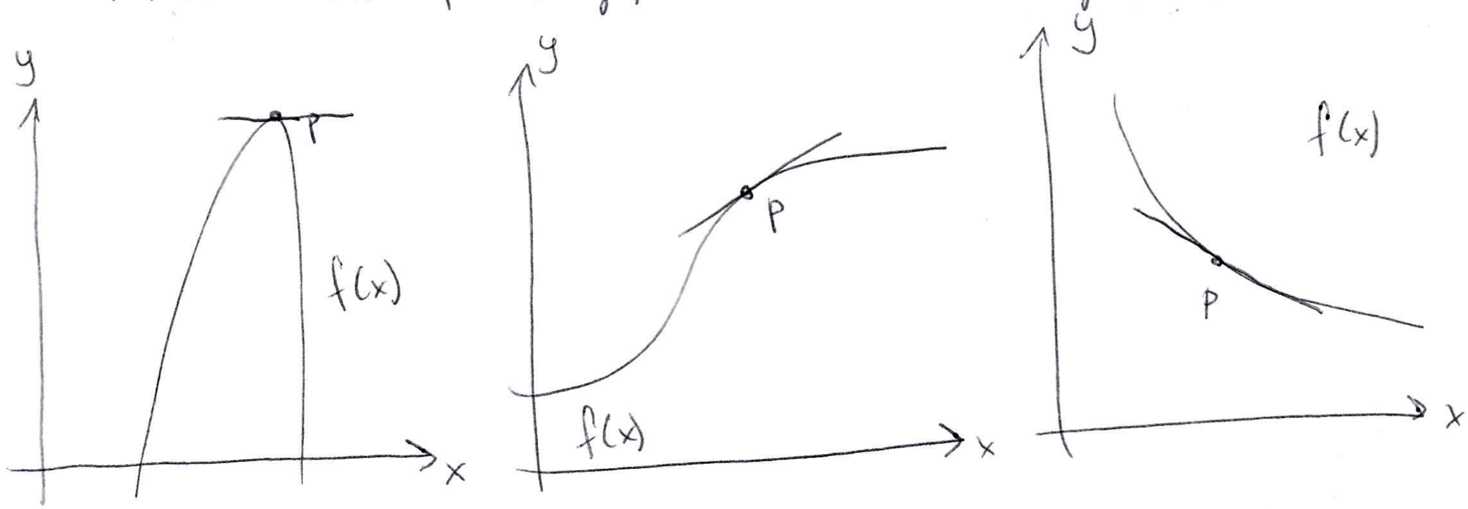
Calculus allows us to study rate of change.

- recall the slope is the rate of change for a line.
- for graphs other than lines, the rate of change is not consistent.



- At (x_1, y_1) the graph rises quickly.
- At (x_2, y_2) the graph rises more slowly
- At (x_3, y_3) the graph levels off
- At (x_4, y_4) the graph is falling.

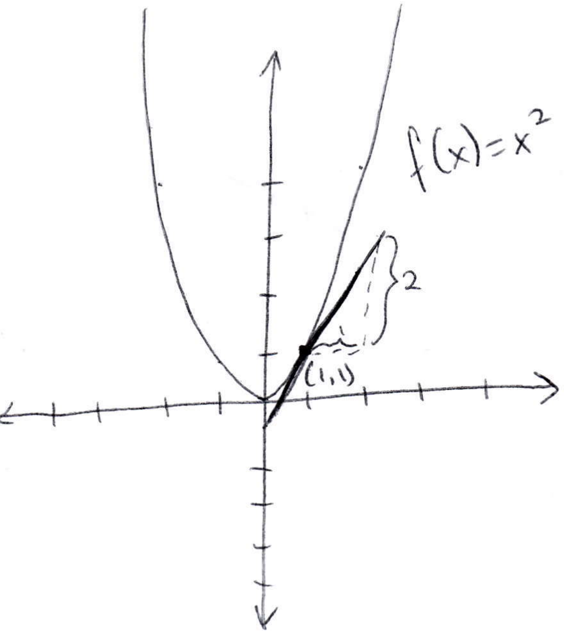
To determine rate of change, we use the tangent line.



The tangent line at a point $P = (x_1, y_1)$ is the line with slope equal to the rate of change of $f(x)$ at P that passes through the point P.

The slope of a graph

To find the slope of a graph at some point, we need to find the slope of the tangent line.



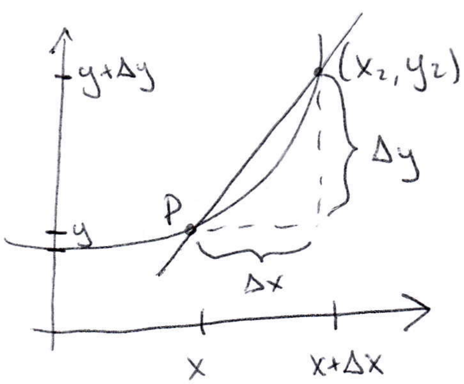
Consider the point $(1,1)$ on $f(x) = x^2$.

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{2}{1} = 2$$

So the graph of $f(x) = x^2$ has a slope of 2 at the point $(1,1)$

Slope and the limit process

A more precise way to estimate the slope of a tangent line uses a secant line. This is a line through the tangent point $P = (x, y)$ and a second point (x_2, y_2) on the graph.



Notice we can write these as

$$P = (x, f(x))$$

and

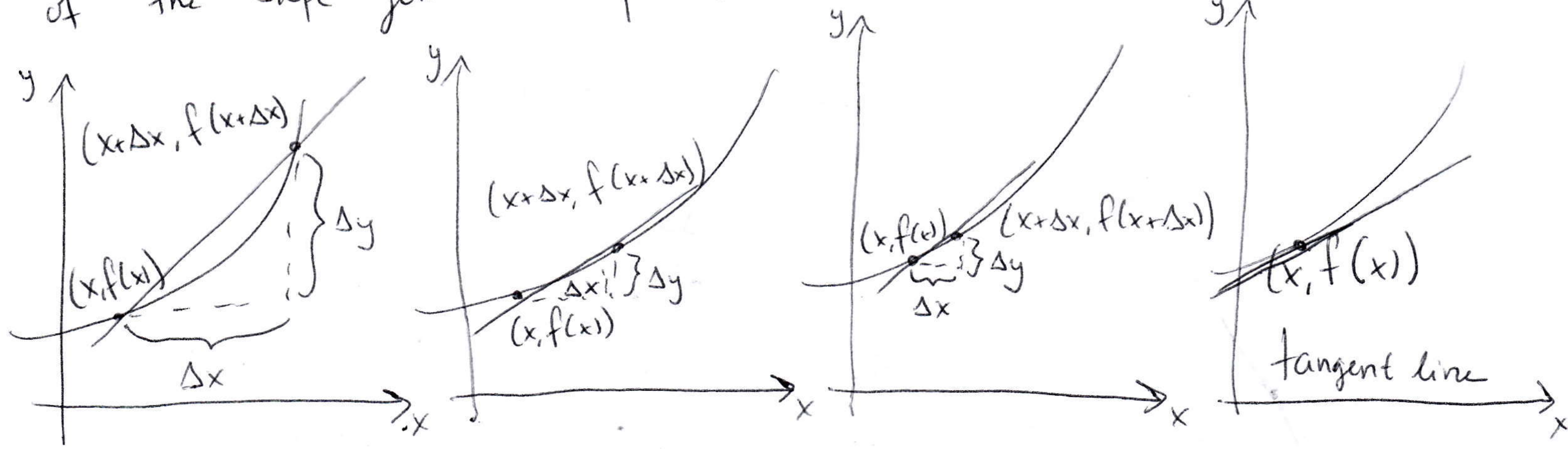
$$(x_2, y_2) = (x + \Delta x, f(x + \Delta x))$$

The slope of the secant line is

$$m_{sec} = \underbrace{\frac{f(x+\Delta x) - f(x)}{\Delta x}}_{\text{"difference quotient"}}$$

Slope and the limit process

By choosing points closer and closer to $(x, f(x))$, the estimation of the slope gets more precise.



As $\Delta x \to 0$, the secant line approaches the tangent line.

Def The slope of a grah:

The <u>slope</u> m of a graph of f at the point $(x, f(x))$ is equal to the slope of the tangent line at $(x, f(x))$. It is given by

$$m = \lim_{\Delta x \to 0} m_{sec} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Ex Find the slope of the graph of $f(x) = x^2$ at the point $(-2,4)$

Soln
$$m_{sec} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \frac{f(-2+\Delta x) - f(-2)}{\Delta x}$$
$$= \frac{(-2+\Delta x)^2 - (-2)^2}{\Delta x}$$
$$= \frac{4 - 4\Delta x + (\Delta x)^2 - 4}{\Delta x}$$
$$= \frac{-4\Delta x + (\Delta x)^2}{\Delta x}$$
$$= -4 + \Delta x, \quad \Delta x \neq 0$$

$$\lim_{\Delta x \to 0} m_{sec} = \lim_{\Delta x \to 0} (-4 + \Delta x)$$
$$= -4$$

So the graph of $f(x) = x^2$ has slope -4 at the point $(-2,4)$.

Ex Find the slope of the graph of $f(x) = -2x + 4$.

Soln We know $m = -2$. Let's confirm using the limit def'n.

$$m = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{[-2(x+\Delta x) + 4] - (-2x+4)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-2x - 2\Delta x + 4 + 2x - 4}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-2\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} -2$$
$$= -2$$

Ex Find a formula for the slope of the graph of $f(x) = x^2 + 1$.

Soln
$$m_{sec} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \frac{[(x+\Delta x)^2 + 1] - (x^2+1)}{\Delta x}$$
$$= \frac{\cancel{x^2} + 2x\Delta x + (\Delta x)^2 \cancel{+1} \cancel{-x^2} \cancel{-1}}{\Delta x}$$
$$= \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$
$$= 2x + \Delta x \qquad \Delta x \neq 0$$

At $(-1, 2)$, the slope is $2(-1) = -2$.

$$\lim_{\Delta x \to 0} m_{sec} = \lim_{\Delta x \to 0} (2x + \Delta x)$$
$$= 2x$$

At $(2, 5)$, the slope is $2(2) = 4$.

The derivative of a function

The derivative of a function is the formula for the slope of a graph. It is denoted $f'(x)$.

Def The derivative of f at x is given by
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
provided this limit exists.

- A function is differentiable at x if its derivative exists at x.
- The process of finding derivatives is called differentiation.

Other notation for $f'(x)$:

$$\frac{dy}{dx} \qquad y' \qquad \frac{d}{dx}[f(x)] \qquad D_x[y].$$

Ex Find the derivative of $f(x) = 3x^2 - 2x$.

Soln
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[3(x+\Delta x)^2 - 2(x+\Delta x)] - (3x^2 - 2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{3x^2} + 6x\Delta x + 3(\Delta x)^2 - \cancel{2x} - 2\Delta x - \cancel{3x^2} + \cancel{2x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x\cancel{\Delta x} + 3(\Delta x)^{\cancel{2}} - 2\cancel{\Delta x}}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} 6x + 3\Delta x - 2$$

$$= 6x - 2$$

Ex Find the derivative of y with respect to t for $y = 2t^{-1}$

Soln
$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{2}{t+\Delta t} - \frac{2}{t}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left(\frac{\frac{2t - 2(t+\Delta t)}{t(t+\Delta t)}}{\Delta t} \right)$$

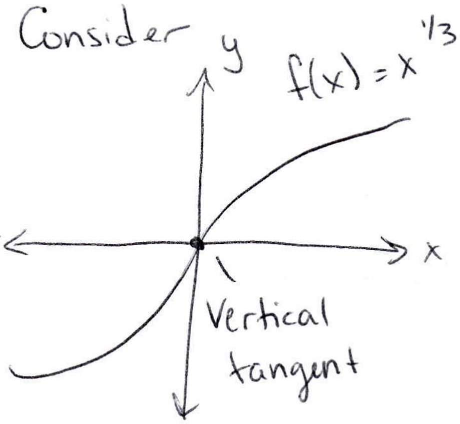
$$= \lim_{\Delta t \to 0} \frac{2t - 2t - 2\Delta t}{t\Delta t(t+\Delta t)}$$

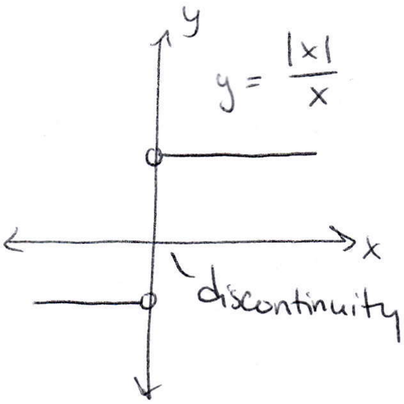
$$= \lim_{\Delta t \to 0} \frac{-2\cancel{\Delta t}}{t\cancel{\Delta t}(t+\Delta t)}$$

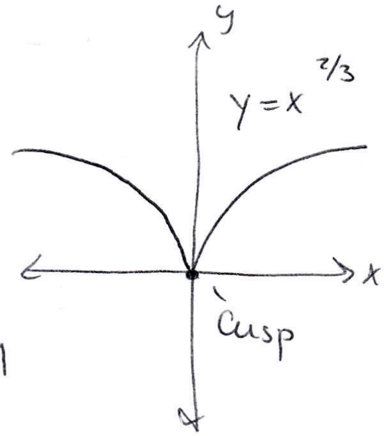
$$= \lim_{\Delta t \to 0} \frac{-2}{t(t+\Delta t)} = \frac{-2}{t^2}$$

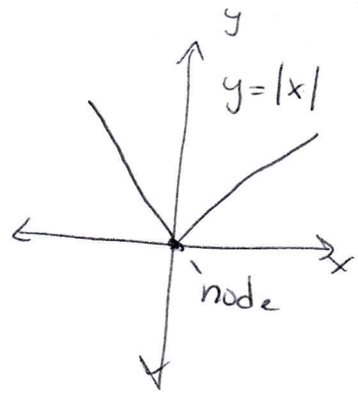
Differentiability and Continuity

Consider









In each graph, the function is differentiable except at $x = 0$.

<u>Theorem</u>: If a function f is differentiable at $x = c$, then f is continuous at $x = c$.