The Constant Rule

The derivative of a constant function is zero.

$$\frac{d}{dx}[c] = 0, \qquad c \text{ is a constant}$$

Proof

Let $f(x) = c$. Then by the limit def'n of a derivative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 0$$

$$= 0$$

So $\frac{d}{dx}[c] = 0$.

Ex $f(x) = \pi$ $\qquad f'(x) = 0$

The Power Rule For n any real number,

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Proof (for n a positive integer) Let $f(x) = x^n$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \dots + (\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \dots + (\Delta x)^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\Delta x + \dots + (\Delta x)^{n-1}\right)$$

$$= nx^{n-1}$$

$\boxtimes$

Notice, when $n=1$

$$\frac{d}{dx}[x] = \frac{d}{dx}[x^1] = 1x^{1-1} = 1x^0 = 1$$

Ex Find the derivatives.

a) $f(x) = x^3$

b) $y = \frac{1}{x^2}$

c) $g(t) = t$

d) $R = x^4$

Soln

a) $f'(x) = 3x^{3-1} = 3x^2$

b) $y = x^{-2} \rightarrow y' = -2x^{-2-1} = -2x^{-3}$

c) $g'(t) = 1$

d) $\frac{dR}{dx} = 4x^3$

Ex Find the slope of the graph of $f(x) = x^2$ for $x = -2, 0, 2$.

Soln $f'(x) = 2x^{2-1} = 2x$

For $x = -2$, $f'(-2) = 2(-2) = -4$

$x = 0$, $f'(0) = 2(0) = 0$

$x = 2$, $f'(2) = 2(2) = 4$

The Constant Multiple Rule

If f is a differentiable function of x and c is a real number, then

$$\frac{d}{dx}[c f(x)] = c \frac{d}{dx}[f(x)] = c f'(x)$$

Proof Recall $\lim_{x \to a} c g(x) = c\left[\lim_{x \to a} g(x)\right]$.

Then consider

$$\frac{d}{dx}[c f(x)] = \lim_{\Delta x \to 0} \frac{c f(x+\Delta x) - c f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} c\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= c \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= c f'(x)$$

$\blacksquare$

Ex $\frac{d}{dx}[5x^2] = 5\frac{d}{dx}[x^2] = 5(2x) = 10x$

$\frac{d}{dx}\left[\frac{x^2}{5}\right] = \frac{1}{5}\frac{d}{dx}[x^2] = \frac{1}{5}(2x) = \frac{2}{5}x$

$f(t) = \frac{4t^2}{5} \qquad f'(t) = \frac{4}{5}\frac{d}{dt}[t^2] = \frac{4}{5}(2t) = \frac{8t}{5}$

Ex $y = \frac{-3x}{2} \qquad y' = \frac{-3}{2}$

$y = 3\pi x \qquad y' = 3\pi$

Ex Using parentheses:

a) $y = \frac{5}{2x^3}$

$y = \frac{5}{2}x^{-3}$

$y' = \frac{5}{2}(-3x^{-4})$

$= \frac{-15}{2}x^{-4}$

b) $y = \frac{5}{(2x)^3}$

$y = \frac{5}{8x^3} = \frac{5}{8}x^{-3}$

$y' = \frac{5}{8}(-3x^{-4})$

$= \frac{-15}{8}x^{-4}$

Ex $\frac{d}{dx}[5x^2] = 5\frac{d}{dx}[x^2] = 5(2x) = 10x$

$\frac{d}{dx}\left[\frac{x^2}{5}\right] = \frac{1}{5}\frac{d}{dx}[x^2] = \frac{1}{5}(2x) = \frac{2}{5}x$

Ex $f(t) = \frac{4t^2}{5} \qquad \frac{d}{dt}\left[\frac{4t^2}{5}\right] = \frac{4}{5}\frac{d}{dt}[t^2] = \frac{4}{5}(2t) = \frac{8}{5}t$

Ex $y = -\frac{3x}{2} \qquad y' = -\frac{3}{2}$

$y = 3\pi x \qquad y' = 3\pi$

Ex (using parentheses)

a) $y = \frac{5}{2x^3}$

$y = \frac{5}{2}x^{-3}$

$y' = \frac{5}{2}(-3x^{-4})$

$= \frac{-15}{2}x^{-4}$

b) $y = \frac{5}{(2x)^3}$

$y = \frac{5}{8x^3} = \frac{5}{8}x^{-3}$

$y' = \frac{5}{8}(-3x^{-4})$

$= \frac{-15}{8}x^{-4}$

Ex $\quad y = \dfrac{1}{2\sqrt[3]{x^2}}$ Find the derivative

Soln
$$y = \frac{1}{2x^{2/3}} = \frac{1}{2}x^{-2/3}$$
$$y' = \frac{1}{2}\left(-\frac{2}{3}x^{-2/3-1}\right)$$
$$= \frac{1}{2}\left(-\frac{2}{3}x^{-5/3}\right)$$
$$= -\frac{1}{3}x^{-5/3}$$

The Sum and Difference Rule

The derivative of the sum or difference of two differentiable functions is given by
$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$
$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$$

Proof let $h(x) = f(x)+g(x)$.
$$h'(x) = \lim_{\Delta x\to 0}\frac{h(x+\Delta x)-h(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{[f(x+\Delta x)+g(x+\Delta x)]-[f(x)+g(x)]}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)+g(x+\Delta x)-g(x)}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]$$
$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}$$
$$= f'(x)+g'(x)$$

We can now differentiate any polynomial!

Ex Find the slope of $f(x) = x^3 - 4x + 2$ at $(1, -1)$

Soln $f'(x) = 3x^2 - 4$

At $(1,-1)$, $f'(1) = 3(1)^2 - 4 = -1$

Ex Find the equation of the tangent line to the graph of
$g(x) = -\frac{1}{2}x^4 + 3x^3 - 2x$ at $(-1, -3/2)$.

Soln $g'(x) = -\frac{1}{2}(4x^3) + 3(3x^2) - 2$
$= -2x^3 + 9x^2 - 2$

At $(-1, -3/2)$, slope is $g'(-1) = -2(-1)^3 + 9(-1)^2 - 2 = 9$

Then $y - y_0 = m(x - x_0)$
$y + 3/2 = 9(x+1)$
$y + \frac{3}{2} = 9x + 9$
$y = 9x + 7.5$

Application

Modeling Social Security Beneficiaries.

From 2000 through 2005, the number of social security beneficiaries can be modeled by

$N = 31.27t^2 + 447.06t + 45412 \quad 0 \le t \le 5$

t is year, $t=0$ is 2000. N is in thousands.

At what rate was the num. of beneficiaries changing in 2002?

Soln $\frac{dN}{dt} = 31.27(2t) + 447.06$
$= 62.54t + 447.06 \quad 0 \le t \le 5$

At $t = 2$
$62.54(2) + 447.06 = 572.14$

Increasing at a rate of 572.14 thousand per year.