The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ a differentiable function of x, then $y = f(g(x))$ is differentiable with respect to x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

OR

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\, g'(x)$$

Ex Find $f(u)$ and $g(x)$ such that $y = f(u) = f(g(x))$.

a) $y = \frac{1}{x+1}$

Soln $y = (x+1)^{-1}$

$g(x) = x+1$

$f(u) = u^{-1}$

b) $y = \sqrt{3x^2 - x + 1}$

Soln $y = (3x^2 - x + 1)^{1/2}$

$g(x) = 3x^2 - x + 1$

$f(u) = u^{1/2}$

Ex Find the derivative of $y=(x^2+1)^3$

Soln $u=g(x)=x^2+1$

$y=f(u)=u^3$

$\frac{dy}{du}=3u^2$ $\qquad$ $\frac{du}{dx}=2x$

$$y'=\frac{dy}{du}\frac{du}{dx}=(3u^2)(2x)$$

$$=3(x^2+1)^2(2x)$$

$$=6x(x^2+1)^2$$

Find the derivative of $y=(x^3+1)^2$

Soln $u=g(x)=x^3+1$

$y=f(u)=u^2$

$\frac{dy}{du}=2u$ $\qquad$ $\frac{du}{dx}=3x^2$

$$y'=\frac{dy}{du}\frac{du}{dx}=(2u)(3x^2)$$

$$=2(x^3+1)(3x^2)$$

$$=6x^2(x^3+1)$$

The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a real number, then

$$\frac{dy}{dx} = n\,[u(x)]^{n-1}\,\frac{du}{dx}$$

OR

$$\frac{d}{dx}\left[u^n\right] = nu^{n-1}\,u'$$

This is a special case of the chain rule!

Ex Find the tangent line to the graph of $y = \sqrt[3]{(x^2+4)^2}$ for $x=2$.

Soln
$$\frac{d}{dx}\Big[(\underbrace{x^2+4}_{u})^{2/3}\Big] = \frac{2}{3}(x^2+4)^{2/3-1}(2x)$$

$$= \frac{4}{3}x(x^2+4)^{-1/3}$$

$$= \frac{4x}{3(x^2+4)^{1/3}}$$

At $x=2$, $y = \sqrt[3]{(4+4)^2} = \sqrt[3]{64} = 4 \rightarrow$ Point $(2,4)$

$$y' = \frac{4(2)}{3\sqrt[3]{4+4}} = \frac{8}{3(2)} = \frac{4}{3} \rightarrow \text{Slope}$$

So the tangent line is
$$y - y_0 = m(x - x_0)$$
$$y - 4 = \frac{4}{3}(x-2)$$
$$y = \frac{4}{3}x + \frac{4}{3}$$

Ex Find the derivatives of a) $y = \frac{3}{x^2+1}$ b) $y = \frac{3}{(x+1)^2}$

a) $y = 3(x^2+1)^{-1}$

$y' = 3\left[-1(x^2+1)^{-2}\right](2x)$

$= -6x(x^2+1)^{-2}$

$= \frac{-6x}{(x^2+1)^2}$

b) $y = 3(x+1)^{-2}$

$y' = 3\left[-2(x+1)^{-3}\right](1)$

$= -6(x+1)^{-3}$

$= \frac{-6}{(x+1)^3}$

Simplification Techniques

Ex Find the derivative of (a) $y = x^2\sqrt{1-x^2}$ (b) $y = \left(\frac{3x-1}{x^2+3}\right)^2$

a) $y = x^2(1-x^2)^{1/2}$

$y' = x^2 \frac{d}{dx}\left[(1-x^2)^{1/2}\right] + (1-x^2)^{1/2}\frac{d}{dx}\left[x^2\right]$

$= x^2\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right] + (1-x^2)^{1/2}(2x)$

$= -x^3(1-x^2)^{-1/2} + 2x(1-x^2)^{1/2}$ Note: $x^2 = x^{1-1/2} = x^1 x^{-1/2}$

$= x(1-x^2)^{-1/2}\left[-x^2 + 2(1-x^2)\right]$

$= x(1-x^2)^{-1/2}(-3x^2+2)$

$= \frac{x(2-3x^2)}{\sqrt{1-x^2}}$

b) $y = \left(\frac{3x-1}{x^2+3}\right)^2$

$$y' = 2\left(\frac{3x-1}{x^2+3}\right)\frac{d}{dx}\left[\frac{3x-1}{x^2+3}\right]$$

$$= \frac{2(3x-1)}{x^2+3}\left[\frac{(x^2+3)(3)-(3x-1)(2x)}{(x^2+3)^2}\right]$$

$$= \frac{2(3x-1)}{x^2+3}\left[\frac{3x^2+9-6x^2+2x}{(x^2+3)^2}\right]$$

$$= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3}$$