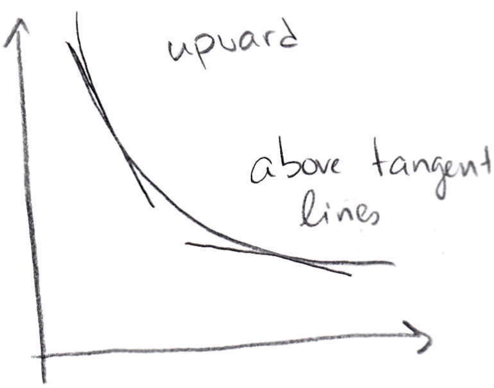
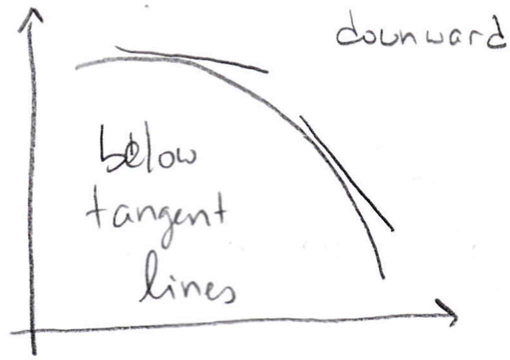
Def Let f be differentiable on an open interval I. The graph of f is

1) Concave upward on I if f' is increasing on I.
2) Concave downward on I if f' is decreasing on I.





Test for concavity:
Let f be a function whose second derivative exists on an open interval I.
1) If $f''(x) > 0$ for all x in I, then f is concave upward on I.
2) If $f''(x) < 0$ for all x in I, then f is concave downward on I.

Applying the test:
1) Find x-values at which $f''(x) = 0$, $f''(x)$ undefined, or f has a discontinuity.
2) Use these values to determine intervals.
3) Test the sign of $f''(x)$ in each interval.

Ex $f(x) = \sqrt{x}$

Soln $f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = -\frac{1}{4}x^{-3/2} < 0$ for $x > 0$ Concave downward

Ex $f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2 > 0$ for all x Concave upward

Ex $f(x) = \frac{6}{x^2+3}$

Soln $f(x) = 6(x^2+3)^{-1}$

$f'(x) = -6(x^2+3)^{-2}(2x)$

$= -12x(x^2+3)^{-2}$

$= \frac{-12x}{(x^2+3)^2}$

$$f''(x) = \frac{(x^2+3)^2(-12) - (-12x)\left[2(x^2+3)(2x)\right]}{(x^2+3)^4}$$

$= \frac{36(x^2-1)}{(x^2+3)^3}$ Defined for all x

$f''(x) = 0$ when $x = \pm 1$

Interval	Test	Sign of f''	Conclusion
$(-\infty, -1)$	-2	$f''(-2) > 0$	Concave upward
$(-1, 1)$	0	$f''(0) < 0$	Concave downward
$(1, \infty)$	2	$f''(2) > 0$	Concave upward

Def If the graph of a continuous function has a tangent line at a point where its concavity changes from upward to downward (or vice versa), then the point is a point of inflection.

Property if $(c, f(c))$ is a point of inflection of f, then either $f''(c) = 0$ or $f''(c)$ is undefined.

Ex $f(x) = x^4 + x^3 - 3x^2 + 1$

Soln $f'(x) = 4x^3 + 3x^2 - 6x$

$f''(x) = 12x^2 + 6x - 6$

$= 6(2x-1)(x+1) \rightarrow f''(x) = 0$ at $x = \frac{1}{2}$, $x = -1$

Interval	Test	Sign	Conclusion
$(-\infty, -1)$	$x = -2$	$+$	Concave upward
$(-1, \frac{1}{2})$	$x = 0$	$-$	Concave downward
$(\frac{1}{2}, \infty)$	$x = 1$	$+$	Concave upward

$\Rightarrow$ points of inflection at $x = -1$ and $x = \frac{1}{2}$

Let $f'(c) = 0$ and let f'' exist on an open interval containing c.

1) if $f''(c) > 0$, $f(c)$ is a relative minimum
2) if $f''(c) < 0$, $f(c)$ is a relative maximum
3) if $f''(c) = 0$ the test fails. Use first derivative test.

Ex $f(x) = -3x^5 + 5x^3$

Soln $f'(x) = -15x^4 + 15x^2$
$= -15x^2(x^2 - 1) \rightarrow f'(x) = 0$ at $x = 0$ and $x = \pm 1$
$f(0) = 0$, $f(-1) = -2$, $f(1) = 2$

$f''(x) = -60x^3 + 30x$
$= -30x(2x^2 - 1)$

Point	Sign of $f''(x)$	Conclusion
$(-1,-2)$	$f''(-1) = 30 > 0$	Relative minimum
$(0,0)$	$f''(0) = 0$	test fails! $\rightarrow$
$(1,2)$	$f''(1) = -30 < 0$	Relative maximum

Interval	Test	Sign $f'(x)$	Concl.
$(-1, 0)$	$-1/2$	+	increase
$(0, 1)$	$1/2$	+	increase

So $(0,0)$ is not a point of relative extrema.