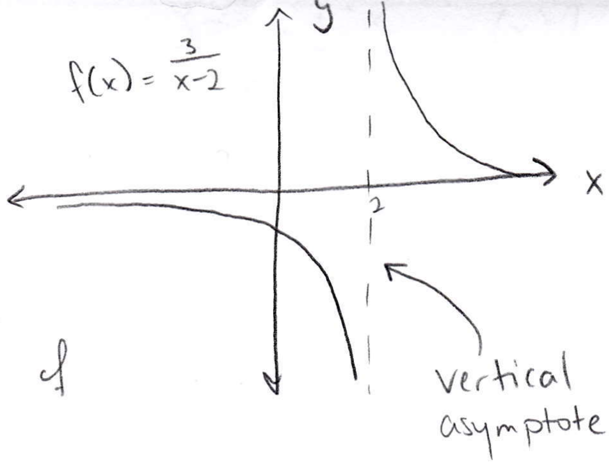
Vertical Asymptotes and Infinite Limits

Consider $f(x) = \frac{3}{x-2}$



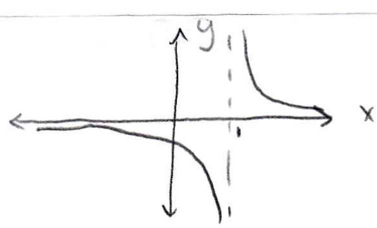
Def If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x=c$ is a vertical asymptote of the graph of f.

Common instance: Rational functions

$$f(x) = \frac{p(x)}{q(x)}$$

where p, q are polynomials. If c is such that $q(x)=0$ and $p(x) \neq 0$, there is a vertical asymptote at $x=c$.

Ex $f(x) = \frac{1}{x-1}$



Asymptote at $x=1$

$\lim_{x \to 1^+} f(x) = \infty$ $\lim_{x \to 1^-} f(x) = -\infty$

Ex $f(x) = \frac{x+2}{x^2-2x}$

Soln $0 = x^2 - 2x$

$= x(x-2)$

$x = 0, 2$

Then $x=0$ and $x=2$ are vertical asymptotes

$(x+2 \neq 0$ for $x=0$ OR $x=2)$.

Ex $f(x) = \dfrac{x^2 + 2x - 8}{x^2 - 4}$

Soln $f(x) = \dfrac{(x+4)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$

$= \dfrac{x+4}{x+2}, \quad x \neq 2$

So there is one vertical asymptote at $x = -2$.

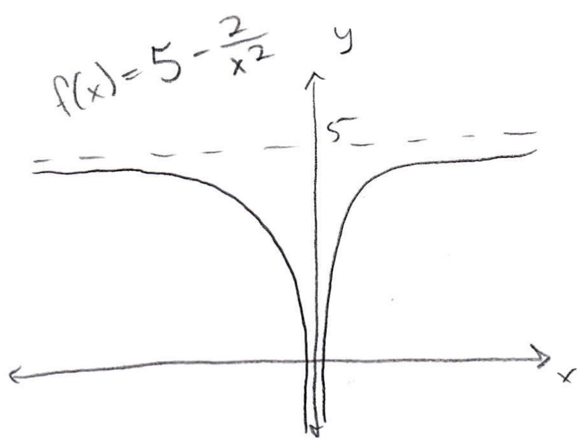
Horizontal Asymptotes and Limits at Infinity

Def If f is a function and L_1 and L_2 are real numbers,

$$\lim_{x \to \infty} f(x) = L_1 \quad \text{and} \quad \lim_{x \to -\infty} f(x) = L_2$$

are limits at infinity. The lines $y = L_1$ and $y = L_2$ are horizontal asymptotes of the graph of f.

Ex $\displaystyle\lim_{x \to \infty} \left(5 - \frac{2}{x^2}\right)$



Soln $\displaystyle\lim_{x \to \infty} \left(5 - \frac{2}{x^2}\right) = \lim_{x \to \infty} (5) - 2 \lim_{x \to \infty} \left(\frac{1}{x^2}\right)$

$= 5 - 0$

$= 5$

So there is a horizontal asymptote at $y = 5$.

Horizontal asymptotes of rational functions:

Let $f(x) = p(x)/q(x)$ be a rational function.

1) If the degree of the numerator $(p(x))$ is less than the degree of the denominator $(q(x))$, then $y=0$ is a horizontal asymptote.

2) If the degree of the numerator is equal to the degree of the denominator, then there is a vertical asymptote $y = \frac{a}{b}$ where a, b are the leading coefficients of $p(x)$, $q(x)$, respectively.

3) If the degree of the numerator is greater than the degree of the denominator, then there are no vertical asymptotes.

Ex $y = \dfrac{-2x+3}{3x^2+1} \rightarrow \dfrac{\text{degree } 1}{\text{degree } 2}$ So $y=0$ is a horizontal asymptote.

Ex $y = \dfrac{-2x^2+3}{3x^2+1} \rightarrow \dfrac{\text{degree } 2}{\text{degree } 2}$ So $y = \dfrac{-2}{3}$ is a horizontal asymptote

Soln $0 = 100 - p$

$p = 100$ is a vertical asymptote.

We can say that cost increases dramatically as the percent p approaches 100%