The Poisson Distribution

August 15, 2019

Midterm Results

Statistic	Raw Score	Percentage
Mean	32.2	71.5
Median	32	71.1
Standard Deviation	5.0	10.8
Maximum	42	93.3
Minimum	19	42.2

Note: Raw scores are out of 45 possible points.
Any extra credit has already been added to the midterm scores.

Calculating Your Current Grade

You can use a little bit of statistics to calculate your current grade!

- Labs are worth 10%
- Quizzes (homeworks) are work 20%
- The midterm is worth 30%
- The final will be worth 40%

Calculating Your Current Grade

- You have grades for $10\% + 20\% = 30\% = 60\%$ of the class.
- We can use this to scale these values and calculate your current overall grade.
  - For your current grade, labs are worth $0.1/0.6 = 0.1667$ or 16.67%
  - Quizzes (homeworks) are worth $.2/.6 = 0.3333$ or 33.33%
  - And the midterm is worth $.3/.6 = 0.50$ or 50%

Calculating Your Current Grade

We can use these to calculate a weighted average. This will give you your current grade.

$$0.167 \times (\text{Lab }\%) + 0.333 \times (\text{Quiz }\%) + 0.5 \times (\text{Midterm }\%)$$

This is just like calculating an expected value!

Calculating Your Current Grade

So if I have 100% in lab, an 85% on quizzes, and got a 70% on the midterm, my current grade is

$$0.167 \times (1) + 0.333 \times (0.85) + 0.5 \times (0.7) = 0.80$$

Note: Remember we drop your lowest lab score! There is also still time to get your quiz grade up a bit.

Midterm Results

- The labs and quizzes are designed to help pad your grade.
- If you're showing up to lab and doing well on your quizzes, I am not worried about your ability to excel in this course.
- If you calculate your grade and it's lower than you want it to be, let's talk!
- If overall grades at the end of the term are low, **I will "curve" the class by adding a set number of points to everyone's grade**.

Working With Cumulative Probabilities

Practice rewriting the following probabilities in terms of $P(X \leq x)$ and $P(X = x_1) + P(X = x_2) + \ldots$.

- $P(X < 4)$
- $P(X > 4)$
- $P(X \leq 4)$
- $P(X \geq 4)$

Working With Cumulative Probabilities

Practice rewriting the following probabilities in terms of $P(X \leq x)$ and $P(X = x_1) + P(X = x_2) + \dots$.

- $P(6 \leq X \leq 8)$
- $P(6 \leq X < 8)$
- $P(6 < X \leq 8)$
- $P(6 < X < 8)$

Motivating Example: Poisson Distribution

- There are about 8 million people in New York City.
- How many New Yorkers would be expect to be hospitalized due to heat attack, each day?
- Historical records suggest the average is 4.4
- But what about the distribution?
- What might a histogram of daily counts look like?

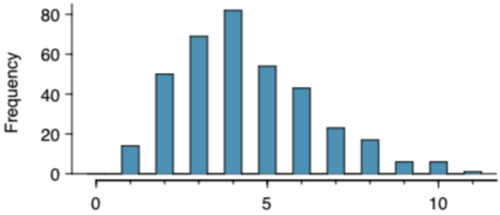
Example

Intuitively, we might think that

- The average is 4.4.
- We don't know the standard deviation.
- The minimum is 0.
- The (theoretical) maximum is about 800 million or so. It's so far away from the mean as to be meaningless.

Clearly the maximum is a lot further from the average than the minimum is, so we might guess that this distribution is skewed to the right.

Example



The number of heart attack hospitalizations were recorded every day for a year.

- The sample mean is 4.38, similar to the historical average of 4.4.
- The sample standard deviation is about 2.
- The distribution is unimodal and right-skewed.

The Poisson Distribution

The **Poisson distribution** is used to describe the number of events that occur in a large population over some period of time. We might measure

- Marriages
- Births
- Heart attacks
- Lightning strikes

In each case, we can count the number of times that event occurs during a period of time.

The Poisson Distribution

- The average number of occurrences per period of time is called the **rate**.
- In the heart attack example, we had a rate of 4.4 *heart attacks* (the event) per *day* (the period of time).
- We denote the rate by the Greek letters λ (lambda) or μ.
- We can use this information to find the probability of observing exactly k events during a particular period of time.

The Poisson Distribution

Suppose we are interested in some events and the number of observed events follows a Poisson distribution with rate λ. Let X be the number of events observed. Then

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

where k can take on any whole number greater than or equal to 0. The letter e is a constant: $e \approx 2.719$.

The Poisson Distribution

The Poisson distribution has an interesting property: its mean and variance are the same!

$$E(X) = Var(X) = \lambda$$

and the standard deviation is then $\sqrt{\lambda}$.

Is It Poisson?

Guidelines for determining if the Poisson distribution is appropriate:

1. We are interested in the number of events that occur.
2. There is a set period of time that we are interested in.
3. Events occur independently of each other.
4. The population that generates the events is quite large.

Example: Coffee Shop Customers

A coffee shop serves an average of 75 customers per hour during the morning rush.

- Which distribution is appropriate for working with the probability of a given number of customers arriving within one hour during the morning rush?

Example: Coffee Shop Customers

1. We are interested in the number of customers served.
   - "Customer served" is the event.
2. We are interested in this event over the course of one hour (during morning rush), so there is a set period of time.
3. We assume that customers are served (more or less) independently of one another.
4. The population that generates these events is anyone who could possibly walk in and be served during the morning rush. That's quite a lot of people, so can be confident that the population is large.

So the Poisson distribution is appropriate.

Example: Coffee Shop Customers

What are the mean and standard deviation of the number of customers this coffee shop serves in one hour during the morning rush?

Example: Coffee Shop Customers

Would it be considered unusually low if only 60 customers showed up to the coffee shop in one hour during the morning rush?

Example: Coffee Shop Customers

Find the probability that the coffee shop serves 70 people in one hour during the morning rush.

Example: Coffee Shop Customers

What is the probability that the coffee shop serves between (and including) 73 and 76 customers?

Poisson Approximation to Binomial

This is another binomial approximation method that will help us avoid difficult factorial expressions.

We can use the Poisson approximation to the binomial distribution when

- n is large
- $np < 7$

Poisson Approximation to Binomial

Suppose we have a lot size of 1000 and the proportion of defective items is 0.001. What is the probability of exactly 3 defective items?

- Items are either defective or not.
- Defective status is independent between items.
- There is a fixed lot size of $n = 1000$ items.
- The probability of success (a defect) is 0.001.

The binomial probability for exactly 3 defectives is

$$\begin{aligned} P(X = 3) &= \binom{1000}{3} (0.001)^3 (0.999)^{1000-3} \\ &= 0.0613 \end{aligned}$$

OR, we can let $\lambda = np = 1$ and use the Poisson distribution:

$$\begin{aligned} P(X = 3) &= \frac{e^{-1} 1^3}{3!} \\ &= 0.0613 \end{aligned}$$