One-Sample Means

August 26, 2019

We can approach hypothesis testing for $\bar{x}$ in mostly the same way that we approached $\hat{p}$. However,

- We will often run into situations where $\bar{x}$ is not approximately normal.
- We will develop a framework for dealing with these situations.

The Central Limit Theorem for $\bar{x}$

When we collect a sufficiently large sample of n independent observations from a population with mean μ and standard deviation σ, the sampling distribution of $\bar{x}$ will be nearly normal with mean

$$\mu$$

and standard error

$$\frac{\sigma}{\sqrt{n}}$$

Conditions for Normality

For the Central Limit Theorem to hold, we require

- Independence
- Normality

Conditions for Normality: Independence

As before, for independence we typically check for

- a simple random sample.
- a random process.

Conditions for Normality: Normality?

For $\bar{x}$ to be normally distributed,

- We require that the observations x used to calculate $\bar{x}$ be from a normally distributed population.
- When this doesn't hold, we require a large sample size.

Large Sample Size

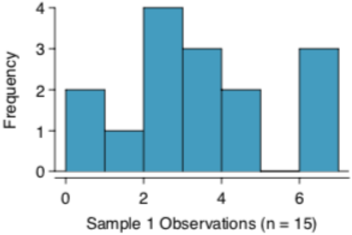
So what is a "large sample size"?

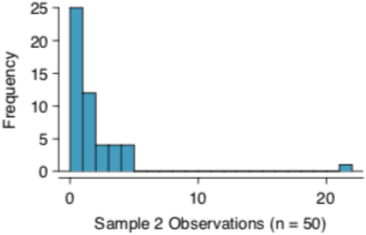
$\boldsymbol{n < 30}$: A sample size n less than 30 is considered a small sample. In this case, we will need the data to come from a nearly normal distribution.

$\boldsymbol{n \geq 30}$: A sample side n of at least 30 is considered a large sample. Then we can assume that $\bar{x}$ is nearly normal.

In both cases, we need to be wary of outliers as these can cause problems with our results.

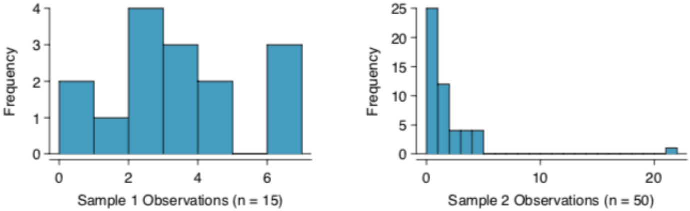
Normal or Not?





Are the independence and normality conditions met in each case?

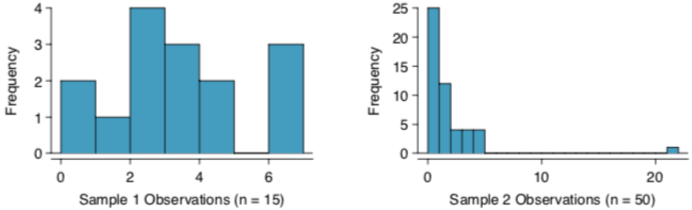
Normal or Not?



Are the independence and normality conditions met in each case?

For the first plot, $n = 15$ and there are no clear outliers. For the mean to be normally distributed, we would have to justify that these are draws from a normal distribution.

Normal or Not?



Are the independence and normality conditions met in each case?

For the second plot, $n = 50$ but there is an extreme outlier at about 22. This outlier will prevent us from confidently using a normal distribution.

One-Sample Means

We will start with the situation wherein we know that $X \sim N(\mu, \sigma)$ and the value of σ is known.

Confidence Interval for μ

This $(1-\alpha)100\%$ confidence interval for μ is

$$\bar{x} \pm z_{\alpha/2} \times \frac{\sigma}{\sqrt{n}}$$

where $\sigma/\sqrt{n}$ is the SE and $z_{\alpha/2}$ is again the critical value.

Example

The following $n = 5$ observations are from a $N(\mu, 2)$ distribution. Find a 90% confidence interval for μ.

$$1.1, \quad 0.5, \quad 2, \quad 1.9, \quad 2.7$$

Example

So we can be 90% confident that the true mean μ is in the interval $(0.1687, 3.1113)$.

Example

Recall that when we say "90% confident", we mean:

- If we draw repeated samples of size 5 from this distribution, then 90% of the time the corresponding intervals will contain the true value of μ.

Confidence Interval for μ

- In practice, we typically do not know the population standard deviation σ.
- Instead, we have to estimate this quantity.
- We will use the sample statistic s to estimate σ.
- This strategy works quite well when $n \geq 30$

Confidence Interval for μ

This works quite well because we expect large samples to give us precise estimates such that

$$SE = \frac{\sigma}{\sqrt{n}} \approx \frac{s}{\sqrt{n}}.$$

Confidence Interval for μ

When $n \geq 30$ and σ is unknown, a $(1 - \alpha)100\%$ confidence interval for μ is

$$\bar{x} \pm z_{\alpha/2} \frac{s}{\sqrt{n}}$$

where we've plugged in s for σ.

Example

The average heart rate of a random sample of 60 students is found to be 74 with a standard deviation of 11. Find a 95% confidence interval for the true mean heart rate of the students.

Sample Size Calculation

Sample size calculation for means is essentially the same as for proportions!

$$n \geq \left(\frac{z_{\alpha/2} \times sd}{MoE} \right)^2$$

Example

A manufacturer claims with 95% confidence that his product is accurate within 0.2 units with a standard deviation of 0.2626. What sample size would we need to demonstrate this claim?

Hypothesis Testing for a Population Mean

We begin with the setting where $n \geq 30$.

- As before, it is possible to use the confidence interval to complete a hypothesis test.
- However, we also want to be able to use our test statistic and p-value approaches.

Hypothesis Testing for a Population Mean

For $n \geq 30$, the test statistic is

$$ts = z = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

where again $s/\sqrt{n} \approx \sigma/\sqrt{n}$ because we are using a large sample.

Hypothesis Testing for a Population Mean

There are five steps to carrying out these hypothesis tests:

1. Write out the null and alternative hypotheses.
2. Calculate the test statistic.
3. Use the significance level to find the critical value

   OR

   use the test statistic to find the p-value.
4. Compare the critical value to the test statistic

   OR

   compare the p-value to α.
5. Conclusion.

Example

In its native habitat, the average density of giant hogweed is 5 plants per m^2. In an invaded area, a sample of 50 plants produced an average of 11.17 plants per m^2 with a standard deviation of 8.9. Does the invaded area have a different average density than the native area? Test at the 5% level of significance.

Hypothesis Testing for a Population Mean

We now move to the situation where $n < 30$.

If $n < 30$ but we are dealing with a normal distribution and σ is known,

$$ts = z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

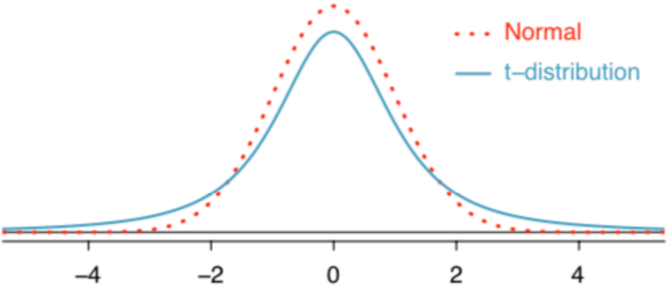
but we know that this will rarely (if ever) occur in practice!

Introducing the t-Distribution

- With a small sample size, plugging in s for σ can result in some problems.
- Therefore less precise samples will require us to make some changes.
- This brings us to the t-distribution.

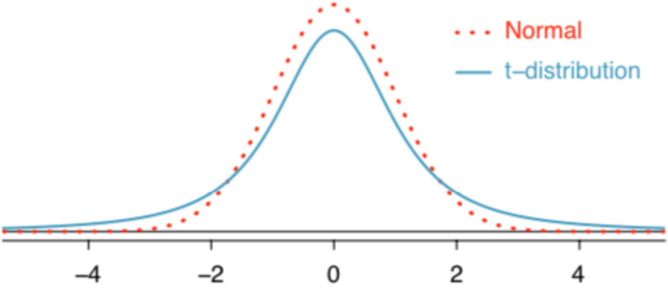
Introducing the t-Distribution

The ***t*-distribution** is a symmetric, bell-shaped curve like the normal distribution.



However, the t-distribution has more area in the tails.

Introducing the t-Distribution



These thicker tails turn out to be exactly the correction we need in order to use s in place of σ when calculating standard error!

The t-Distribution

The t-distribution:

- Is always centered at zero.
- Has one parameter: degrees of freedom (df).
- For our purposes,

$$df = n - 1$$

where n is our sample size.

The t-Distribution

- For the t-distribution, the parameter df controls how fat the tails are.
- Higher values of df result in thinner tails.
  - That is, higher df results in a t-distribution that looks more like a normal distribution.
  - I.e., bigger sample sizes make the t-distribution look more normal.
- When $n \geq 30$, the t-distribution will be essentially equivalent to the normal distribution.

Confidence Intervals for A Single Population Mean

When $n < 30$ and σ is unknown, we use the t-distribution for our confidence intervals. A $(1-\alpha)100\%$ confidence interval for μ is

$$\bar{x} \pm t_{\alpha/2,df} \times \frac{s}{\sqrt{n}}$$

Critical Values for the t-Distribution

Let's take a minute to look at the table of t-distribution critical values that will be provided for your final exam.

Test Statistics

The test statistic for the setting where $n < 30$ and σ is unknown is

$$ts = t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

(two-sided hypotheses)

P-Values

The p-value for the two-sided hypotheses is then

$$2 \times P(t_{df} < -|ts|)$$

Example

The following data is on red blood cell counts (in 10^6 cells per microliter) for 9 people:

$$5.4, \quad 5.3, \quad 5.3, \quad 5.2, \quad 5.4, \quad 4.9, \quad 5.0, \quad 5.2, \quad 5.4$$

Test at the 5% level of significance if the average cell count is 5.