Factorial Experiments

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Cautionary Comments on Blocking

- When designating one factor as a **block**, we assume that the treatment will have the same effect, regardless of block used.
- When the factors interact, we need a new experimental design setting.

Example

The manager of a manufacturing plant suspects that production line output depends on

1. which of two supervisors is in charge.
2. which of three shifts it is.

Interactions

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- There's an *interaction* whenever there is a relationship between the two factors.
- Example: Supervisor 1 may be a night owl and perform best at night, while Supervisor 2 tends to doze off during night shifts.
- Essentially, different levels of `shift` impact the two supervisors differently.

Example

Each supervisor is observed on three randomly selected days for each of the three shifts.

	Shift		
Supervisor	Day	Swing	Night
1	487	498	550
2	602	602	637

Example

Now suppose we got the following data instead:

	Shift		
Supervisor	Day	Swing	Night
1	602	498	450
2	487	602	657

Factorial Experiments

The previous example is one of a **factorial experiment**.

- There are $2 \times 3 = 6$ treatments (factor level combinations).
- This is called a **2 × 3 factorial experiment**.
- We can also use factorial experiments to look at more than two factors and their interactions.

Replication

- In a factorial experiment, we want multiple observations per treatment.
- These are called **replications**.
- E.g., we could take three data points at each factor level combination.
- We will assume that each treatment is replicated r times.

ANOVA for an $a \times b$ Factorial Experiment

We will use the following notation:

- a levels of factor A
- b levels of factor B
- r replicates of each of the ab factor combinations
- A total of $n = abr$ observations

Sum of Squares for an $a \times b$ Factorial Experiment

We now partition our variance into four parts:

$$\text{SS Total} = \text{SSA} + \text{SSB} + \text{SS(AB)} + \text{SSE}$$

- SSA measures variation among factor A means.
- SSB measures variation among factor B means.
- SS(AB) measures variation among the different combinations of factor levels.
- SSE measures the variation within each combination of factor levels (experimental error).

Sum of Squares for an $a \times b$ Factorial Experiment

- We refer to SSA and SSB as the **main effect** sums of squares.
- SS(AB) is referred to as the **interaction** sum of squares.

Degrees of Freedom for an $a \times b$ Factorial Experiment

Each source of variation has an accompanying degrees of freedom:

- $df_{\text{A}} = a - 1$
- $df_{\text{B}} = b - 1$
- $df_{\text{AB}} = (a - 1)(b - 1)$
- $df_{\text{error}} = ab(r - 1)$
- $df_{\text{total}} = n - 1 = abr - 1$

The mean square for each source of variation is the sum of squares divided by its degrees of freedom.

ANOVA: Randomized Block Design

Source	df	SS	MS	F
A	$a-1$	SSA	MSA $=\frac{SSA}{a-1}$	$\frac{MSA}{MSE}$
B	$b-1$	SSB	MSB $=\frac{SSB}{b-1}$	$\frac{MSG}{MSE}$
AB	$(a-1)(b-1)$	SS(AB)	MS(AB)$=\frac{SS(AB)}{(a-1)(b-1)}$	$\frac{MS(AB)}{MSE}$
Error	$ab(r-1)$	SSE	$MSE=\frac{SSE}{ab(r-1)}$	
Total	$abr-1$	SSTotal		

Tests for a Factorial Experiment

For the main effect of factor A:

H_0 : No differences among the factor A means.

H_A : At least two of the factor A means differ.

Compare:

$$F = \frac{MSA}{MSE} \quad \text{to} \quad F_\alpha(df_1 = a - 1, df_2 = ab(r - 1)).$$

Tests for a Factorial Experiment

For the main effect of Factor B:

$$\begin{aligned} H_0 : &\quad \text{No differences among the factor B means.} \\ H_A : &\quad \text{At least two of the factor B means differ.} \end{aligned}$$

$$F = \frac{MSB}{MSE} \quad \text{to} \quad F_\alpha(df_1 = b - 1, df_2 = ab(r-1)).$$

Tests for a Factorial Experiment

For the interaction of factors A and B:

$$H_0: \quad \text{Factors A and B do not interact.}$$
$$H_A: \quad \text{Factors A and B interact.}$$

Compare

$$F = \frac{MS(AB)}{MSE} \quad \text{to} \quad F_\alpha(df_1 = (a-1)(b-1), df_2 = ab(r-1)).$$

Example

The two supervisors were monitored on three randomly selected days for each of the three shifts:

	Shift		
Supervisor	Day	Swing	Night
	571	480	470
1	610	474	430
	625	540	450
	480	625	630
2	516	600	680
	465	581	661

Example: Exploratory Analysis

We might want to examine the data for possible interactions. This table shows the means across each set of replicates:

	Shift		
Supervisor	Day	Swing	Night
	571	480	470
1	610	474	430
	625	540	450
Mean	**602**	**498**	**450**
	480	625	630
2	516	600	680
	465	581	661
Mean	**487**	**602**	**657**

Example

For two supervisors monitored on three randomly selected days for each of three shifts,

- SSA= 19208 (supervisor)
- SSB= 247 (shift)
- SS(AB)= 81127 (interaction)
- SSE= 8640
- SSTotal= 109222

Finish the ANOVA table for these data.