Goal: derive least squares estimates for $y = \beta_0 + \beta_1 x + \epsilon$.

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{f}(x_i))^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

Step 1: Derivatives wrt $\hat{\beta}_0$ and $\hat{\beta}_1$

$$\begin{aligned}
\frac{\delta}{\delta \hat{\beta}_0} \left(\sum_{i=1}^{n} e_i^2 \right) &= \sum_{i=1}^{n} \frac{\delta}{\delta \hat{\beta}_0} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \\
&= \sum_{i=1}^{n} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-1) \\
&= -2 \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) \\
\frac{\delta}{\delta \hat{\beta}_1} \left(\sum_{i=1}^{n} e_i^2 \right) &= \sum_{i=1}^{n} \frac{\delta}{\delta \hat{\beta}_1} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \\
&= \sum_{i=1}^{n} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-x_i) \\
&= -2 \sum_{i=1}^{n} (x_i y_i - \hat{\beta}_0 x_i - \hat{\beta}_1 x_i^2)
\end{aligned}$$

Step 2: Set both derivatives equal to zero and simplify.

$$\begin{aligned}
0 &= -2 \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) \\
&= \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) \\
&= \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \hat{\beta}_0 - \sum_{i=1}^{n} \hat{\beta}_1 x_i \\
&= n\bar{y} - n\hat{\beta}_0 - \hat{\beta}_1 n \bar{x} \\
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x}
\end{aligned}$$

and

$$\begin{aligned}
0 &= -2 \sum_{i=1}^{n} (x_i y_i - \hat{\beta}_0 x_i - \hat{\beta}_1 x_i^2) \\
&= \sum_{i=1}^{n} (x_i y_i - \hat{\beta}_0 x_i - \hat{\beta}_1 x_i^2) \\
&= \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} \hat{\beta}_0 x_i - \sum_{i=1}^{n} \hat{\beta}_1 x_i^2 \\
&= \sum_{i=1}^{n} x_i y_i - \hat{\beta}_0 \sum_{i=1}^{n} x_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2
\end{aligned}$$

then

$$
\begin{aligned}
\hat{\beta}_1 \sum_{i=1}^{n} x_i^2 &= \sum_{i=1}^{n} x_i y_i - \hat{\beta}_0 \sum_{i=1}^{n} x_i \\
&= \sum_{i=1}^{n} x_i y_i - (\bar{y} - \hat{\beta}_1 \bar{x}) \sum_{i=1}^{n} x_i \\
&= \sum_{i=1}^{n} x_i y_i - \frac{1}{n} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i - \hat{\beta}_1 \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\beta}_1 \sum_{i=1}^{n} x_i^2 - \hat{\beta}_1 \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right)^2 &= \sum_{i=1}^{n} x_i y_i - \frac{1}{n} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \\
\hat{\beta}_1 \left[\sum_{i=1}^{n} x_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right)^2 \right] &= \sum_{i=1}^{n} x_i y_i - \frac{1}{n} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i
\end{aligned}
$$

So then

$$
\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right)^2}
$$